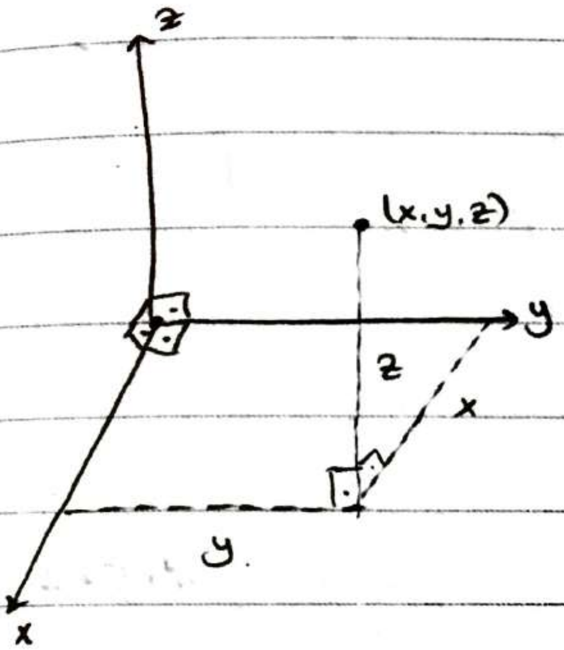
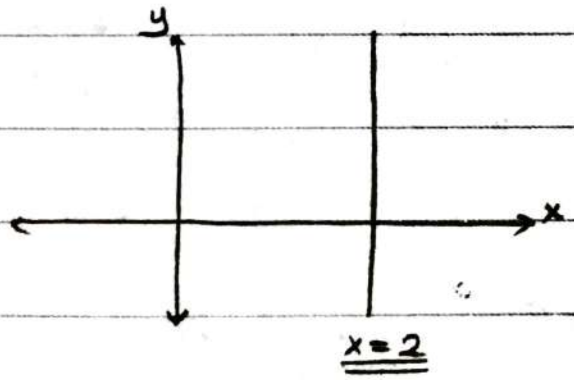
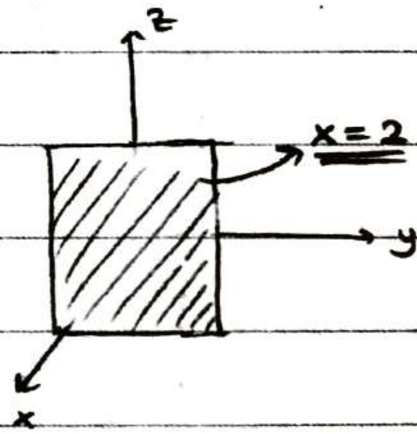
CHAPTER 12 - VECTORS AND GEOMETRY OF SPACES

12.1 3D COORDINATES

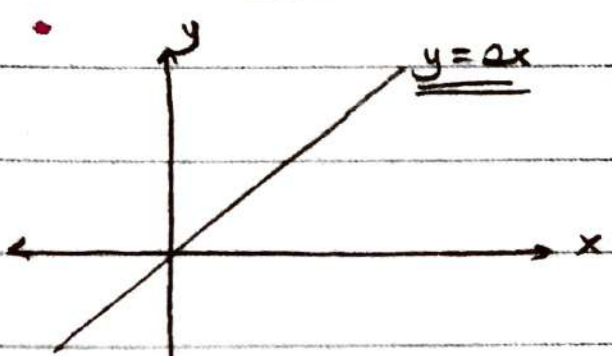


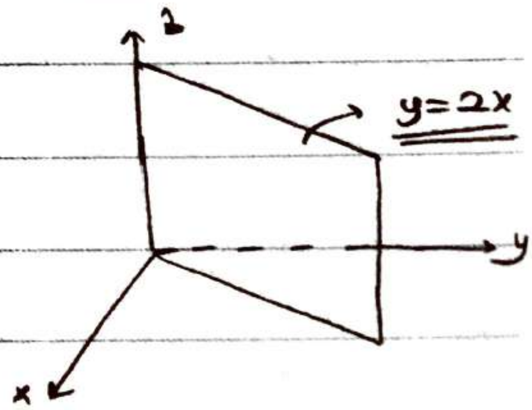
ex: $\mathbb{R}^2$ $\mathbb{R}^3$



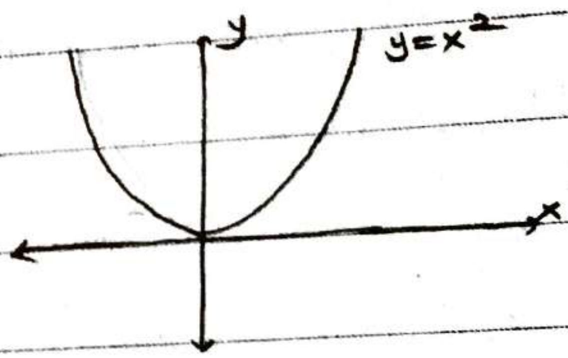


$\mathbb{R}^2$ $\mathbb{R}^3$

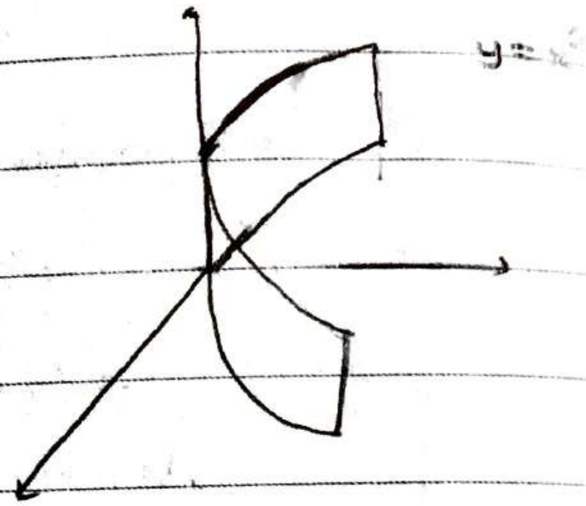




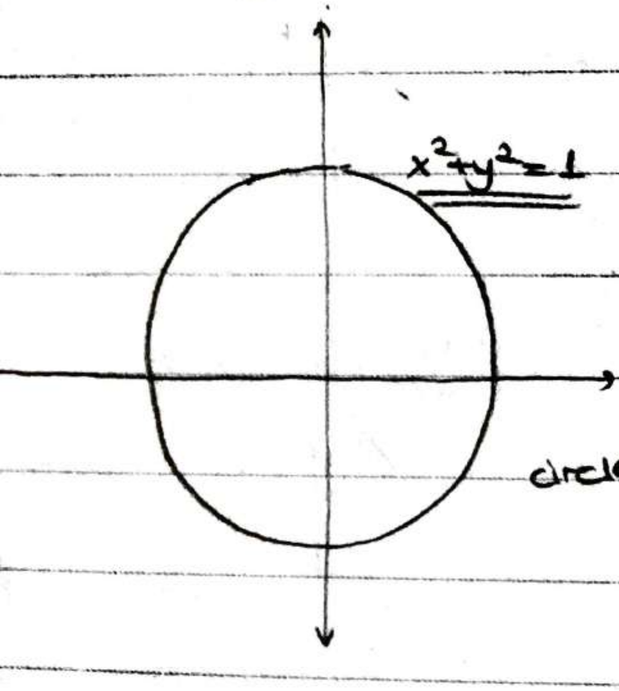
$\mathbb{R}^2$



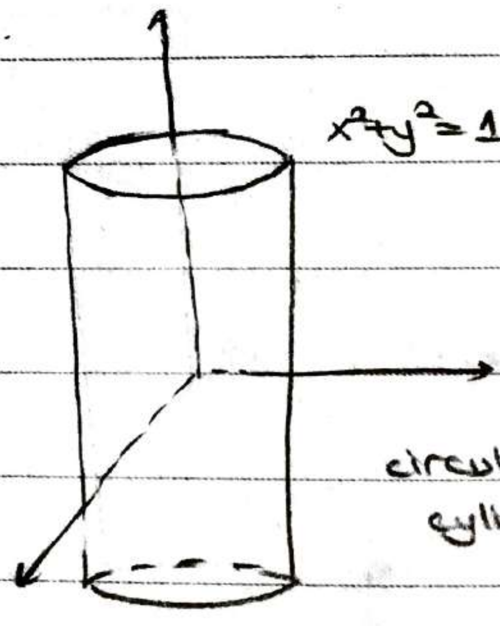
$\mathbb{R}^3$

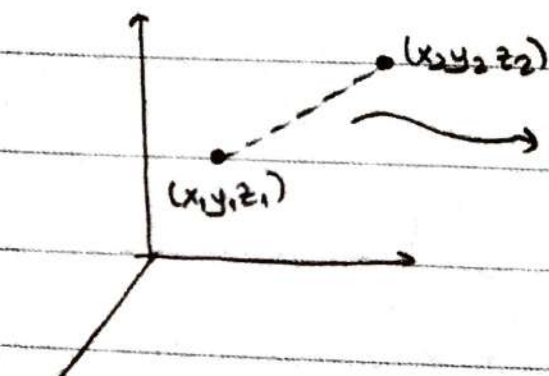


$\mathbb{R}^2$



$\mathbb{R}^3$





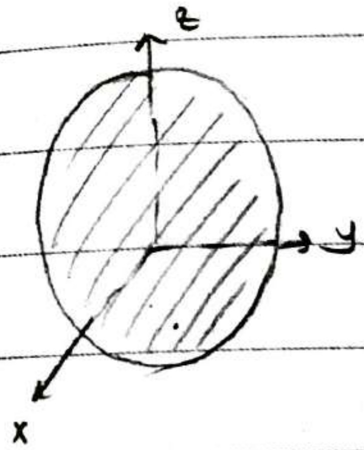
$$\text{distance} = \sqrt{(x^2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

sphere $P(x_0, y_0, z_0)$ point

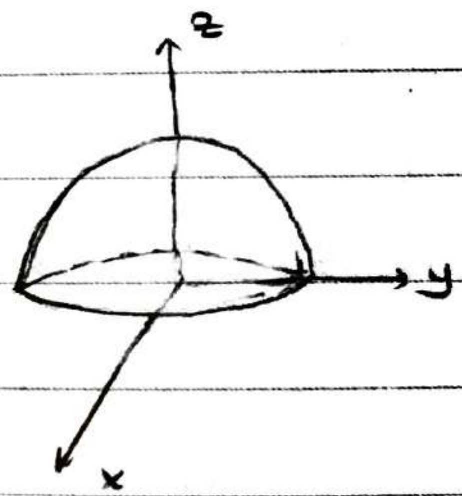
All the points equidistant to P is a sphere whose equation is

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$$

ex1 $x^2+y^2+z^2 < 2^2$ = interior of the sphere with center $(0,0,0)$ and radius = 2

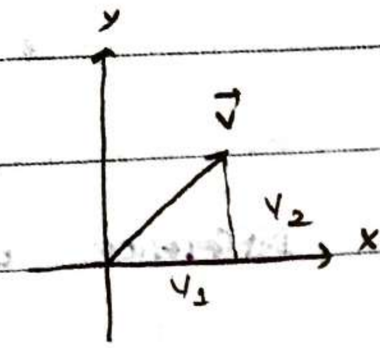


ex: $x^2+y^2+z^2=4$ $\quad z \geq 0$

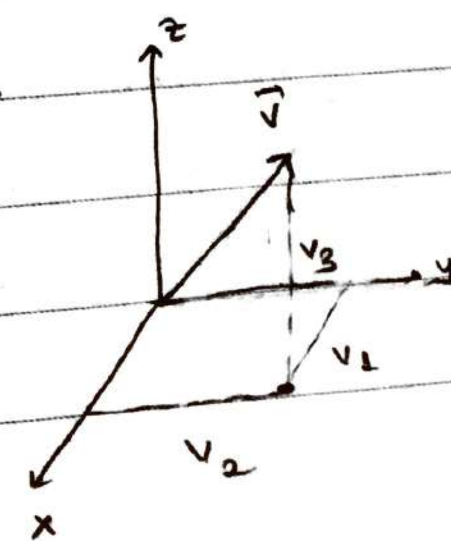


12.2 VECTORS

2-vector $\vec{v} = \langle v_1, v_2 \rangle$



3-vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$



zero vector $\vec{0} = \langle 0, 0, 0 \rangle$

vector addition

$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\vec{u} + \vec{v} = \langle u_1 + v_1 , u_2 + v_2 , u_3 + v_3 \rangle$

scalar multipication

$k \in \mathbb{R}$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$k . \vec{v} = \langle k . v_1 , k v_2 , k v_3 \rangle$

$|k\vec{v}| = |k| . |\vec{v}|$

proof:

$|k\vec{v}| = \sqrt{(kv_1)^2 + (kv_2)^2 + (kv_3)^2}$

$= |k| \sqrt{v_1^2 + v_2^2 + v_3^2} = |k| . |\vec{v}|$

Difference of vectors

$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$

$\langle u_1 , u_2 , u_3 \rangle - \langle v_1 , v_2 , v_3 \rangle$

ex1 $\vec{u} = \langle -1, 3, 1 \rangle$

$\vec{v} = \langle 4, 7, 0 \rangle$

i) $2\vec{u} - 3\vec{v}$

$= 2\langle -1, 3, 1 \rangle - 3\langle 4, 7, 0 \rangle$

$= \langle -2, 6, 2 \rangle - \langle 12, 21, 0 \rangle$

$= \langle -14, -15, 2 \rangle$

ii) $|2\vec{u} - 3\vec{v}| = (14^2 + 15^2 + 2^2)^{1/2}$

i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

ii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

iii) $\vec{u} + \vec{0} = \vec{u}$

iv) $\vec{u} + (-\vec{u}) = \vec{0}$

v) $0\,\vec{u} = \vec{0}$

vi) $1.\vec{u} = \vec{u}$

vii) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

viii) $(a+b)\vec{u} = a\vec{u} + b\vec{u}$

Unit Vectors

$\vec{u}$ = unit vector if $|\vec{u}| = 1$

$\mathbb{R}^3 \rightarrow i = \langle 1, 0, 0 \rangle$

$j = \langle 0, 1, 0 \rangle$

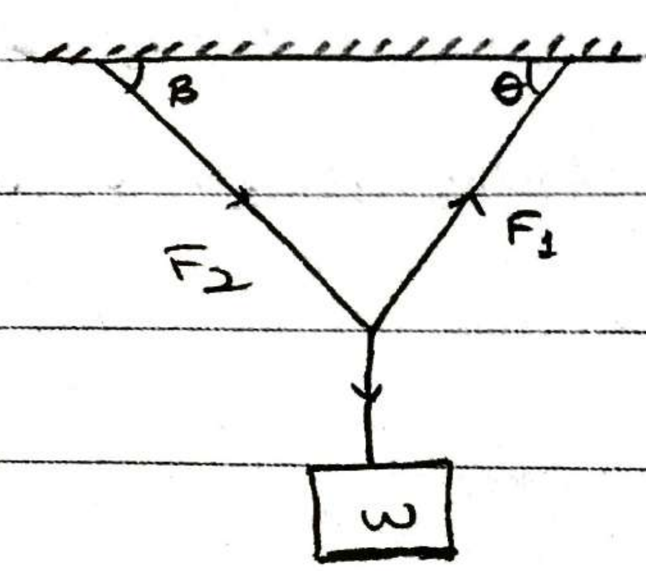
$k = \langle 0, 0, 1 \rangle$

ex: Find a unit vector $\vec{u}$ in the direction of the vector from $P(1,0,1)$ to $Q(3,2,0)$

$$\vec{PQ} = \langle 3-1,\ 2-0,\ 0-1 \rangle = \langle 2,2,-1 \rangle$$

$$\vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{\langle 2,2,-1 \rangle}{\sqrt{2^2+2^2+1^2}} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{-1}{3} \right\rangle$$

ex:



$\vec{F_1} = ?$

$\vec{F_2} = ?$

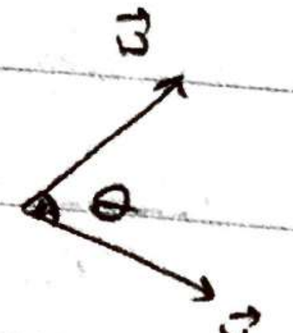
12.3 DOT PRODUCT

Definition: $\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

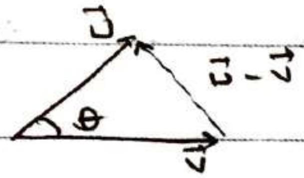
dot product $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

Theorem:



$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$

proof:



$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$$

$$\vec{u} = \langle u_1, u_2, u_3 \rangle$$

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$|\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}}$$

$$|\vec{u}|^2 = \vec{u}\cdot\vec{u}$$

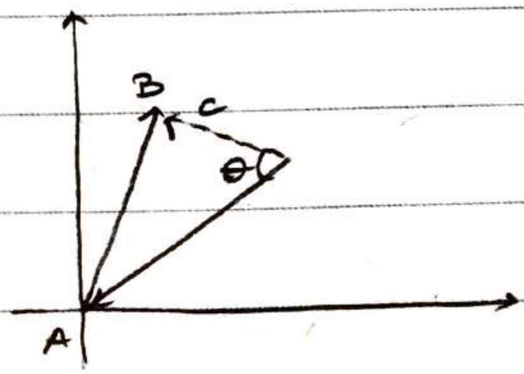
$$u_1^2 + u_2^2 + u_3^2 \qquad u_1^2 + u_2^2 + u_3^2$$

$$(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2$$

$$u_1^2 + v_1^2 + u_2^2 + v_2^2 + u_3^2 + v_3^2 - 2u_1v_1 - 2u_2v_2 - 2u_3v_3$$

$$|\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u}\,\vec{v}$$

ex: $A(0,0)$, $B(3,5)$, $C(5,2)$ Find the angle between $\vec{CA}$ and $\vec{CB}$



$$\vec{CA} = -5\hat{i} - 2\hat{j}$$

$$\vec{CB} = (3-5)\hat{i} + (5-2)\hat{j} = -2\hat{i} + 3\hat{j}$$

$$\vec{CA}\,\vec{CB} = |\vec{CA}|\,|\vec{CB}|.\cos\theta$$

$$= (5^2 + 2^2)^{1/2}(2^2 + 3^2)^{1/2}\cos\theta$$

$$\cos\theta = \frac{4}{\sqrt{29}.\sqrt{13}}$$

$$\theta = \arccos\frac{4}{\sqrt{29}\sqrt{13}} \approx 78.1^\circ$$

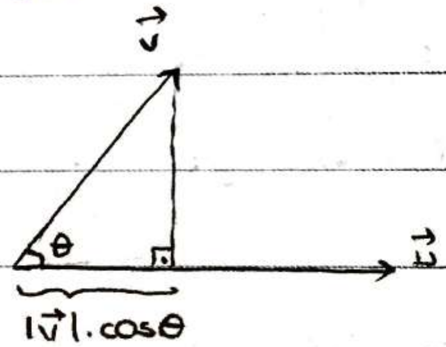
Definition! Two vectors $\vec{u}, \vec{v}$ are $\underbrace{\text{orthogonal}}_{\text{dik}}$

if $\vec{u} \cdot \vec{v} = 0$

properties of dot product

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $(c.\vec{u})\vec{v} = \vec{u}(c\vec{v}) = c(\vec{u}\vec{v})$
3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u}\vec{v} + \vec{u}\vec{w}$
4. $\vec{u}\,\vec{u} = |\vec{u}|^2$
5. $\vec{0} \cdot \vec{u} = 0$

projection



$$\text{proj}_{\vec{u}}\vec{v} = \underbrace{(|\vec{v}|\cos\theta)}_{\text{magnitude}} \cdot \underbrace{\frac{\vec{u}}{|\vec{u}|}}_{\text{direction}}$$

$$= \frac{|\vec{u}||\vec{v}|\cos\theta}{|\vec{u}|} \frac{\vec{u}}{|\vec{u}|}$$

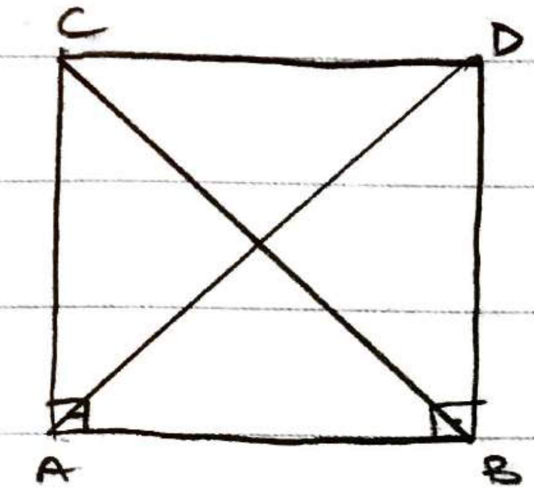
$$\text{proj}_{\vec{u}}\vec{v} = \frac{|\vec{u}\vec{v}|}{|\vec{u}|} \frac{\vec{u}}{|\vec{u}|}$$

ex: Find the vector projection of

$\vec{v} = 6\hat{i} + 3\hat{j} + 2k$ onto $\vec{u} = \hat{i} - 2\hat{j} - 2k$

$$\text{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{u}.\vec{v}}{|\vec{u}|^2}\right) \cdot \vec{u} = \frac{6-6-4}{1+4+4}\,(\hat{i} - 2\hat{j} - 2k)$$

ex: Show that squares are only rectangules with perpendicular diagonals.



$\vec{AD} \cdot \vec{BC} = 0$

$(\vec{AB} + \vec{BD})(\vec{BD} + \vec{DC}) = 0$

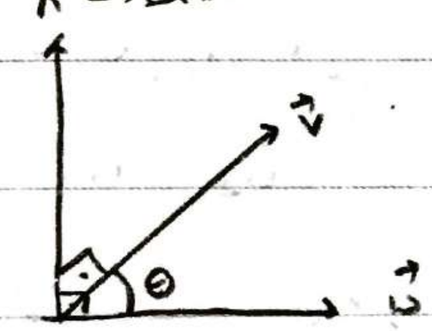
$\vec{AB} \cdot \vec{BD} + \vec{AB}\,\vec{DC} + \vec{BD}\,\vec{BD} + \vec{BD}\,\vec{DC} = 0$

$0 - |\vec{AB}|^2 + |BD|^2 + 0 = 0$

$|AB| = |BD| \Rightarrow ABCD$ is a square.

14.4 CROSS PRODUCT

$\vec{n} \rightarrow$ unit vector



$\vec{u} \cdot \vec{v} = \underbrace{(|\vec{u}||\vec{v}| \sin\theta)}_{\text{magnitude}} \vec{n}$ ↳ direction

if $\vec{u}$ an $\vec{v}$ are parallel

$\vec{u} \times \vec{v} = \vec{0}$

and if $\vec{u} \times \vec{v} = 0 \Rightarrow \vec{u}$ and $\vec{v}$ are parallel

properties

1) $(r\vec{u}) \times (s\vec{v}) = (rs)(\vec{u} \times \vec{v})$

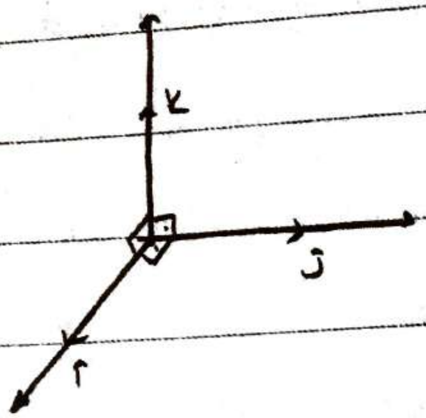
2) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

3) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

4) $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$

5) $0 \times \vec{u} = \vec{0}$

6) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$



$$\hat{i} \times \hat{j} = k$$

$$\hat{j} \times k = \hat{i}$$

$$k \times \hat{i} = \hat{j}$$

$$\vec{u} = u_1 i + u_2 j + u_3 k$$

$$\vec{v} = v_1 i + v_2 j + v_3 k$$

$$\vec{u} \times \vec{v} = u_1 v_1 \; i \times i + u_1 v_2 \; i \times j + u_1 v_3 \; i \times k +$$

$$u_2 v_1 \; j \times i + u_2 v_2 \; \hat{j} \times j + u_2 v_3 \; j \times k$$

$$u_3 v_1 \; k \times i + u_3 v_2 \; k \times j + u_3 v_3 \; k \times k$$

$$= (u_2 v_3 - u_3 v_2) i - (u_1 v_3 - u_3 v_1) j + (u_1 v_2 - u_2 v_1) k$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

$$= \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

ex: Find $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ if

$\vec{u} = 2i + j + k$, $\vec{v} = -4i + 3j + k$

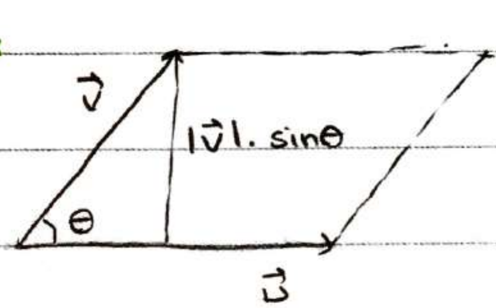
$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$$

$$= i \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - j \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + k \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}$$

$$= (1-3)i - (2+4)j + (6+4)k$$

$$= -2i - 6j + 10k$$

$$\vec{v} \times \vec{u} = -\vec{u} \times \vec{v} = 2i + 6j - 10k$$



$$\text{Area} = \underbrace{|\vec{u}|}_{\text{base length}}\underbrace{|\vec{v}|.\sin\theta}_{\text{height}} = |\vec{u} \times \vec{v}|$$

ex: Find the area of the parallelogram spanned by

$\vec{u} = 2i + j + k$

$\vec{v} = 4i + 3j + k$

$$\text{Area} = |\vec{u} \times \vec{v}| = |-2i - 6j + 10k|$$

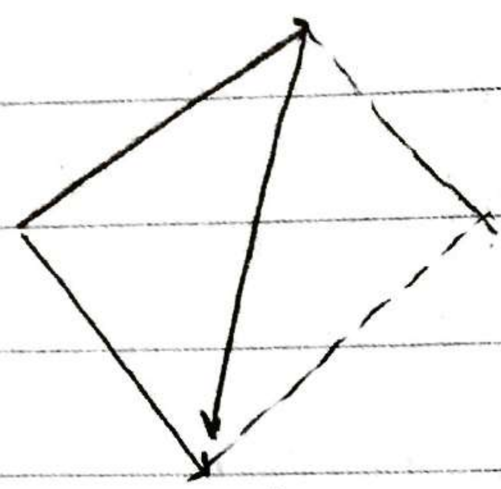
$$= \sqrt{4 + 36 + 100} = \sqrt{140}$$

ex: $P(1,-1,0)$

$B(2,1,-1)$ $\qquad$ $\text{AREA}(P\hat{Q}R) = ?$

$\downarrow$

$R(-1,1,2)$ $\qquad$ $\frac{1}{2}\cdot|PQ \times PR|$



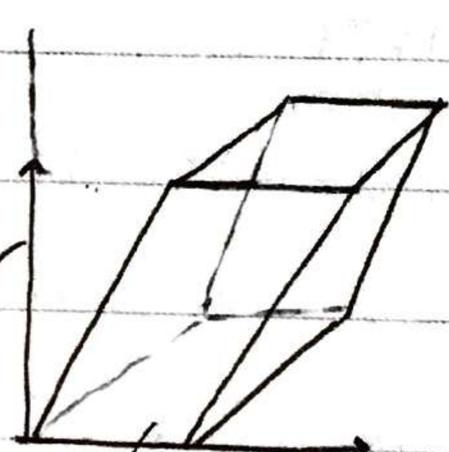
$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = i\begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}$$

$$= (4+2)i - (2-2)j + (2+4)k$$

$$= 6i + 6k$$

$$\text{Area} = \frac{1}{2}\sqrt{6^2+6^2} = 3\sqrt{2}$$

ex:



Base Area $= |\vec{u} \times \vec{v}|$

$$\text{proj}_{\vec{u}\times\vec{v}}\,\vec{w} = \left(\frac{\vec{w}\cdot(\vec{u}\times\vec{v})}{|\vec{u}\times\vec{v}|^2}\right)\vec{u}\times\vec{v}$$

$$\text{height} = |\text{proj}_{\vec{u}\times\vec{v}}\,\vec{w}| = \frac{|\vec{w}\,(\vec{u}\times\vec{v})|}{|\vec{u}\times\vec{v}|}$$

$$\text{volume} = \text{Base Area} * \text{height}$$

$$= |\vec{u} \times \vec{v}| \; \frac{|\vec{w} \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

$$\text{Volume} = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$$

ex: Find the volume of the box determined by

$\vec{u} = i + 2j - k$

$\vec{v} = -2i + 3k$

$\vec{w} = 7j - 4k$

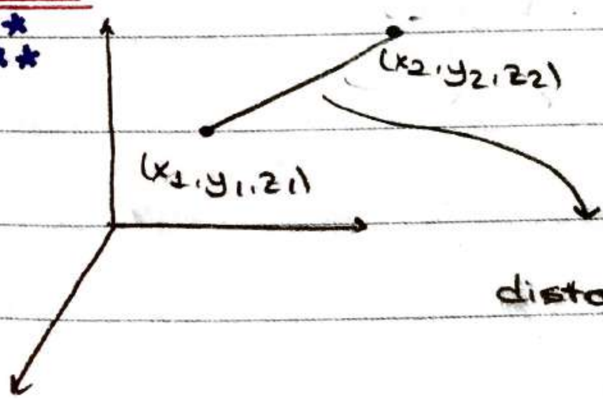
$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} k$$

$$= 6i - 3j + 4k$$

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = (6i + 4k) \cdot (7j - 4k) = 6.0 - 1.7 - 4.4$$

$$= -23$$

Recall!



$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$